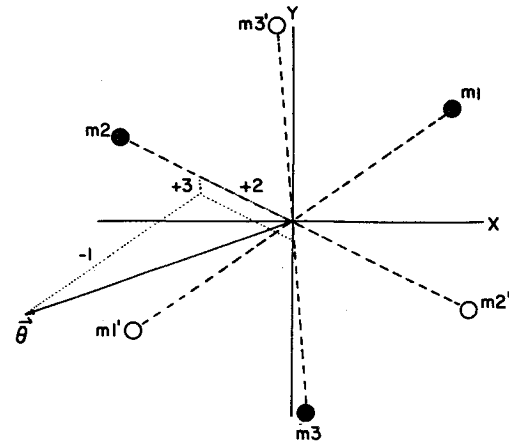

Fig. 3 Optimum wobble removal using three masses.

correspond with the local minimum θ obtainable by movement of a single mass. It is possible from the vector diagram, however, to find where the local minimum should occur and to use this fact while monitoring the wobble removal. It is also apparent from the diagram how the phase ψ should vary during the maneuver, and to periodically check its progress. In practice, it is more difficult to measure ψ than θ, so it is useful to notice that arrival at a local minimum provides an evaluation of the phase that is independent of the error in the initial value unless the error is such that the wobble vector is placed in a wrong sector. In this case, there could be an ambiguity of 180° in the phase indicated by the local minimum θ, which should be easy to resolve experimentally.

In practice it has been possible to reduce θ from initial values of approximately 0.25° to less than 0.02°, which has proved satisfactory for the Telesat domestic satellite communications system.

References

[1] McIntyre, J. E. and Gianelli, M. J., "Bearing Axis Wobble for a Dual Spin Vehicle," *Journal of Spacecraft and Rockets*, Vol. 8, No. 9, Sept. 1971, pp. 945–951.

Probability of Crashing for a Terrain-Following Missile

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Nomenclature

A = event of a positive crossing of level h_o, event of a crash
B = event of one or more crashes in time τ_T
$E(t), e(t)$ = missile altitude error random process, sample function
$E'(t), e'(t)$ = missile altitude error rate random process, sample function
$\mathrm{erf}(x)$ = error function of $x = \frac{1}{(2\pi)^{1/2}} \int_0^x \exp(-z^2/2)\,dz$
f_T' = terrain bandwidth, cycles/naut mile
f_T = terrain bandwidth, cycles/sec = $Vf_T'/6076.2$
$h(t), h_{CL}(t), h_o$ = missile altitude, clearance altitude, commanded altitude, mean clearance altitude
$H_T(t), h_T(t)$ = terrain altitude random process, sample function
L_T = correlation length of terrain, naut miles
N = order of terrain
N_S = order of missile altitude control system equivalent Butterworth form
n = number of terrain cycles = $Rf_T' = Tf_T$
$P\{\ \}$ = probability of event $\{\ \}$
P_C = probability of crashing = $1 - P_{NC}$
$P_C^{(A)}, P_C^{(B)}$ = probability of crashing computed by method A, method B
P_{NC} = probability of not crashing
p = probability of missile being on or below ground level = $\frac{1}{2} - \mathrm{erf}(h_o/\sigma_e)$
q = $1 - p$
R = range, naut miles
s = $j\omega$
T = flight time, sec
V = missile speed, fps
$W_p(s)$ = transfer function relating particular missile response p to terrain input h_T
θ = missile/terrain bandwidth ratio = ω_b/ω_T
λ, λ_B = average frequency of event A, event B
ν = np
$\sigma_e^2, \sigma_{e'}^2, \sigma_h^2, \sigma_T^2$ = variance of altitude error, error rate, altitude, terrain
τ = time shift, sec
τ_e, τ_T = correlation time for altitude error, terrain, sec
$\Phi_{xx}(s)$ = power spectral density of terrain (for deviations about the mean)
$\phi_{ee}(\tau)$ = autocorrelation function of altitude error
Ω = $\sigma_{e'}/\sigma_e$
ω = frequency variable, rad/sec
ω_b, ω_T = bandwidth of missile altitude control system, terrain, rad/sec

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Index category: LV/M Mission Studies and Economics.
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Introduction

An important criterion in evaluating the performance of a low-flying missile in a terrain-following mode is its crash probability (probability of "clobber") over a certain range or flight time due to terrain variations. The missile is assumed to have an altitude control system with no forward-looking radar, thereby lessening its chance of detection. A linearized model of the altitude control system is used in conjunction with a statistical model of the terrain to determine P_C or its complement P_{NC}. The main value of such a linearized study is to compare performance trends and missile-terrain interactions for various configurations.

The expression commonly used for P_C by investigators of the terrain-following problem (for example, see Refs. 1 and 2 and at Cornell Aeronautical Lab., Buffalo, N.Y.) is derived by dividing the total range into a number of statistically-independent segments each of length equal to the correlation length of the terrain. There are shortcomings in this method as discussed below. In this Note, a new expression is derived for P_C and proposed as a replacement in circumstances where the assumptions upon which it is based are valid. In general, it gives more pessimistic results, i.e., the missile must fly higher in order to achieve a P_{NC} equal to that obtained by the older method.

Terrain-Following

The missile is assumed to be controlled to a certain altitude h_o above the terrain, flying at constant speed V, as indicated in Fig. 1a. A statistical model of the terrain is required for analytical studies of the terrain-following problem. Any terrain sample over which the missile may fly is regarded as a realization

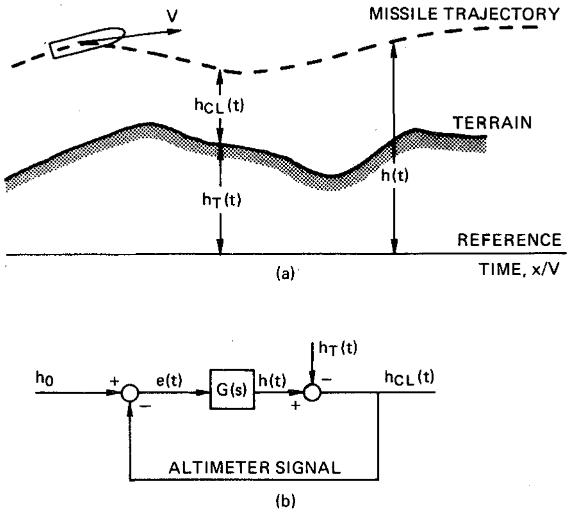

Fig. 1 Terrain-following.

of a stationary ergodic random process $H_T(t)$ with spectral density (for deviations about the mean) in the Butterworth form

$$\Phi_{xx}(s) = \frac{K_N}{(-1)^N (s/\omega_T)^{2N} + 1} \tag{1}$$

thereby explicitly exhibiting the terrain variance σ_T^2 and bandwidth ω_T which are the main terrain parameters of interest in this problem. For example, $K_1 = 2\sigma_T^2/\omega_T$, $K_2 = 2(2)^{1/2}\sigma_T^2/\omega_T$, $K_3 = 3\sigma_T^2/\omega_T$. Work at Cornell Aeronautical Lab. indicated that, with $N > 1$, Eq. (1) adequately represented many of the terrain samples studied.

The Whittle two-dimensional terrain model[3,4] with spectral density of the form

$$\Phi_{xx}(\omega) = \frac{K}{(1+(\omega/\omega_T)^2)^2}, \qquad \omega = (\omega_x^2 + \omega_y^2)^{1/2} \tag{2}$$

may be written as

$$\Phi_{xx}(s) = \frac{4\sigma_T^2 \omega_T^3}{(s + 2\omega_T s + \omega_T^2)(s^2 - 2\omega_T s + \omega_T^2)} \tag{3}$$

Thus, it bears a close resemblance to the $N = 2$ case and can be treated similarly.

Since the terrain serves as a random input to the linearized time-invariant missile altitude control system (Fig. 1b), the various responses such as missile altitude, angle of attack, acceleration, etc., are also stationary random processes with variances given by

$$\sigma_p^2 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} W_p(s) W_p(-s) \Phi_{xx}(s)\, ds \tag{4}$$

where $W_p(s)$ is the transfer function relating the particular response variable p to the terrain input $h_T(t)$. Integrals of the form (4) are evaluated symbolically in Ref. 5 for example. Of particular interest are σ_e^2 and $\sigma_{e'}^2$, the variances of altitude error and error rate, respectively. These are obtained from Eq. (4) by replacing $W_p(s)$ by $W_e(s)$ and $sW_e(s)$.

A typical missile altitude loop transfer function $W_h(s) = G(s)/[1+G(s)]$ would be, at least, fifth-order with two zeroes, one in the right-half plane. However, it can be shown[6] that when $W_h(s)$ is replaced by a Butterworth form of suitable order with the same bandwidth, the resulting variances as computed by Eq. (4) are very close to those obtained with the realistic transfer function. The order N_s of the equivalent Butterworth system will be called the order of the system. As a result, a key parameter in linearized statistical studies of terrain-following is the ratio of missile bandwidth to terrain bandwidth rather than the actual value of either. This can be seen for the simple cases shown in Table 1 and can be extended to more complex missile-terrain combinations. The altitude loop is assumed to have zero steady-state error. It will be noted that since $W_e(s) = 1/[1+G(s)]$, $\sigma_{e'}^2$ is not defined by Eq. (4) unless $N > 1$. One approach in practice, for the $N = 1$ case, is to write

$$\Phi_{xx}(s) = \frac{K}{(s+\omega_T)(-s+\omega_T)} \approx \frac{K}{(\delta s^2 + s + \omega_T)(\delta s^2 - s + \omega_T)} \tag{5}$$

with $0 < \delta \ll 1$, δ being selected at the largest value which will permit the above approximation to hold within the accuracy of the analysis. Then a large, but finite, value of $\sigma_{e'}$ may be obtained from Eq. (4) for the approximation.

Usual Method for Computing P_C

The expression commonly used for computing P_C is

$$P_C = 1 - (1-p)^n \tag{6}$$

This is based on the assumption that the altitude error process $E(t)$ is ergodic and normally distributed. Equation (6) may be derived from reasoning that the total flight time T is divided into n independent intervals during each of which the event of success (not crashing) is equiprobable with probability $q = 1 - p$. Each interval is of length $L_T = 1/f_T'$ naut miles or $\tau_T = 1/f_T$ sec. Thus, for a first-order terrain, the correlation time τ_T would be the time shift between samples for the terrain autocorrelation function to fall to $e^{-2\pi} \approx 0.002$ of σ_T^2 (correlation times for higher order processes will be defined similarly). At such time, the terrain samples are essentially uncorrelated and if $H_T(t)$ is normal, as will be assumed, they are independent also. There are two drawbacks with the preceding approach: 1) from the ergodic hypothesis, p may be regarded as the proportion of the time that the missile is on or below the ground during any flight time T. Strictly speaking

$$p = \lim_{T \to \infty} T(h_o)/T \tag{7}$$

where $T(h_o)$ is the time that $e(t) \geqq h_o$ during T. Even if T is not infinite in Eq. (7), it should be at least long enough to justify the assumption of ergodicity. This fact is ignored. 2) It is evident that it is the correlation time τ_e of $E(t)$ rather than of $H_T(t)$ which should be of concern since it is the behavior of $E(t)$ relative to the arbitrary level h_o which determines whether the missile crashes. Thus, the estimate of P_C provided by Eq. (6) would be more accurate if $n = T/\tau_e$ rather than using the value defined in the nomenclature.

The preceding method for deriving P_C will be called method B. It will be noted that P_C, being the complement of P_{NC}, the probability of not crashing, will be the probability of one *or more* crashes in time T. The proposed new technique for deriving P_C will be called method A and is developed in the next section.

Method A for Computing P_C

A sample function $e(t)$ of the error random process $E(t)$ might be as shown in Fig. 2a. The missile hits the ground when

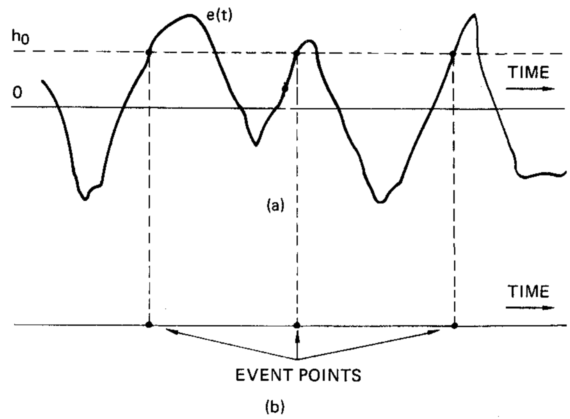

Fig. 2 Crossings of arbitrary level by altitude error signal.

Table 1 Missile response variances as functions of missile/terrain bandwidth ratio

Order of altitude control system N_s	1			2		3
Terrain order N	1	2	3	1	2	1
$\frac{\sigma_e^2}{\sigma_T^2}$	$\frac{1}{1+\theta}$	$\frac{1}{1+(2)^{1/2}\theta+\theta^2}$	$\frac{2+\theta}{2[1+2\theta+2\theta^2+\theta^3]}$	$\frac{1+[3/(2)^{1/2}]\theta}{1+(2)^{1/2}\theta+\theta^2}$	$\frac{1+3\theta+2\theta^2}{1+2\theta+2\theta^2+2\theta^3+\theta^4}$	$\frac{3+8\theta+10\theta^2}{3[1+2\theta+2\theta^2+\theta^3]}$
$\frac{\sigma_{e'}^2}{\omega_T^2\sigma_T^2}$	...	$\frac{1+(2)^{1/2}\theta}{1+(2)^{1/2}\theta+\theta^2}$	$\frac{1+2\theta}{2[1+2\theta+2\theta^2+\theta^3]}$	...	$\frac{1+2\theta+4\theta^2+3\theta^3}{1+2\theta+2\theta^2+2\theta^3+\theta^4}$	...
$\frac{\sigma_{e'}^2}{\omega_T^2\sigma_e^2}$	...	$1+(2)^{1/2}\theta$	$\frac{1+2\theta}{2+\theta}$	...	$\frac{1+2\theta+4\theta^2+3\theta^3}{1+3\theta+2\theta^2}$	...
$\frac{\sigma_h^2}{\sigma_T^2}$	$\frac{\theta}{1+\theta}$	$\frac{\theta[(2)^{\frac{1}{2}}+\theta]}{1+(2)^{1/2}\theta+\theta^2}$	$\frac{\theta[3+4\theta+2\theta^2]}{2[1+2\theta+2\theta^2+\theta^3]}$	$\frac{\theta\{[1/(2)^{1/2}]+\theta\}}{1+(2)^{1/2}\theta+\theta^2}$	$\frac{\theta[1+2\theta+2\theta^2+\theta^3]}{1+2\theta+2\theta^2+2\theta^3+\theta^4}$	$\frac{\theta[2+4\theta+3\theta^2]}{3[1+2\theta+2\theta^2+\theta^3]}$

$e(t)$ crosses the h_o line with $e'(t) > 0$. This will be called a positive crossing. Consider each such crossing to be an event A as depicted in Fig. 2b. The mean frequency λ of these events may be computed as for the classical "zero-crossing" problem treated in many probability texts.[7,8] Under the assumptions that $E(t)$ is stationary and normally distributed, we obtain the well-known result

$$\lambda(h_o) = \frac{1}{2\pi}\frac{\sigma_{e'}}{\sigma_e}\exp\left[-\frac{h_o^2}{2\sigma_e^2}\right] \tag{8}$$

for the average frequency of a positive crossing or a crash. The parameter $\Omega = \sigma_{e'}/\sigma_e$ is a function of ω_T and bandwidth ratio θ as implied by Table 1. For $N = 3$, as θ increases, $\Omega \rightarrow \omega_T(2)^{1/2}$ and this holds for higher values of N_s also. For fixed ratio, h_o/σ_e, λ is thus proportional to ω_T in this case. For $N = 2$, however, under the same conditions, λ is seen to vary as $\theta^{1/2}$ as well which is somewhat unexpected.

The mean frequency of events A, does not, per se, indicate the probability distribution in time of these events. We must consider this further for the present application. The error correlation time τ_e may be determined from the autocorrelation function

$$\phi_{EE}(\tau) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} W_e(s)W_e(-s)\Phi_{xx}(s)\,e^{s\tau}\,ds \tag{9}$$

For a zero steady-state error system, $W_e(s)$ will have the form $sQ(s)/R(s)$ with $sQ(s)$, in general, being of the same order in s as $R(s)$. Because of the differentiation, $E(t)$ will be, in general, a more rapidly fluctuating random process than the terrain process $H_T(t)$. For $N_s = N = 1$

$$\tau_e \approx [\ln \theta/2\pi(\theta-1)]\tau_T < \tau_T \tag{10}$$

if $\theta > 1$. For $N_s = 2$, $N = 1$

$$\frac{\phi_{EE}(\tau)}{\phi_{EE}(o)} = \frac{\theta^2+(2)^{1/2}\theta+1}{(3/(2)^{1/2}\theta+1)}\left[\frac{-(2\theta^2-1)}{\theta^4+1}e^{-\omega_T\tau} + \frac{\theta(5)^{1/2}}{(\theta^4+1)^{1/2}}e^{-a\tau}\cos(b\tau-\psi)\right] \tag{11}$$

where $a = b = \omega_T\theta/(2)^{1/2}$ and $\psi = \tan^{-1}[(\theta^2-3)/(3\theta^2+1)]$. It may be verified numerically for this and other cases where $N_s \geq N$ and $\theta > 1$ (as is true in general) that τ_e will be less than τ_T and that the ratio τ_e/τ_T decreases as θ increases.

Returning to Fig. 2b, events A separated by a time interval of τ_e or greater may be considered to be statistically independent. We will make the following assumptions; 1) the total flight time T is large compared with τ_e; and 2) the commanded altitude h_o is large enough (for example, $h_o \geq 3\sigma_e$) so that the resulting value of $\lambda(h_o)$ as given by Eq. (8) would imply that the probability of more than one event A in an interval of length τ_e is negligible.

Under these assumptions, we can consider the number of events A in time T to be Poisson-distributed with mean frequency λ, i.e.,

$$P\{k \text{ crashes in time } T\} = (\lambda T)^k/k!\exp(-\lambda T) \tag{12}$$

It follows that the probability of not hitting the ground in T sec is

$$P_{NC} = \exp(-\lambda T) \tag{13}$$

so that the probability of one or more crashes in T sec is

$$P_C = 1 - \exp\left[-\frac{1}{2\pi}\frac{\sigma_{e'}}{\sigma_e}\exp\left(\frac{-h_o^2}{2\sigma_e^2}\right)T\right] \tag{14}$$

A Comparison between the Two Methods

Using Eq. (6), P_{NC} by method B, called $P_{NC}^{(B)}$, can be put in a form similar to Eq. (13) by assuming that, comparatively speaking, for a long time of flight with $h_o \geq 3\sigma_e$, the number n of independent intervals becomes large and p is small while the product $v = np$ is of moderate magnitude. Then

$$P_{NC}^{(B)} = (1 - v/n)^n \tag{15}$$

so that

$$\ln P_{NC}^{(B)} = n\ln(1 - v/n) = -v - v^2/2n - \cdots \tag{16}$$

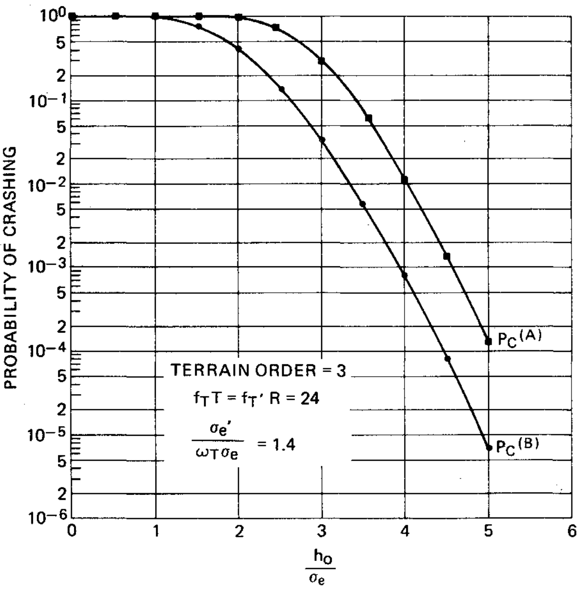

Fig. 3 Comparison of crash probability vs commanded altitude.

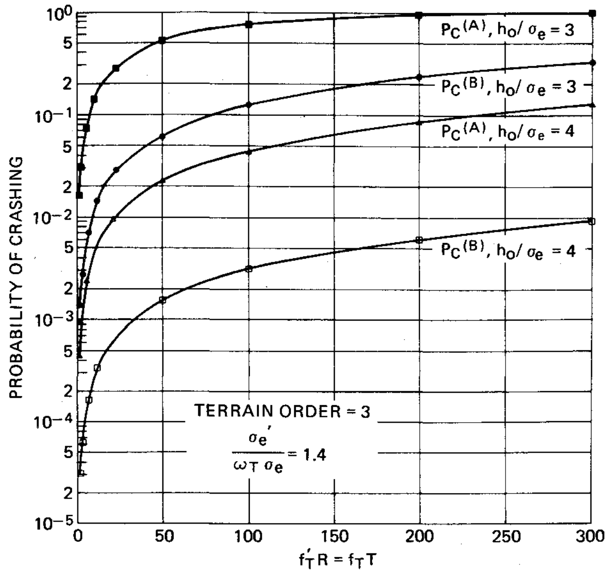

Fig. 4 Comparison of crash probability vs number of terrain cycles.

Thus, for large n

$$P_{NC}^{(B)} \approx \exp(-\nu) = \exp(-\lambda_B T) \tag{17}$$

where

$$\lambda_B = (\omega_T/2\pi)\left[\tfrac{1}{2} - \operatorname{erf}(h_o/\sigma_e)\right] \tag{18}$$

Strictly speaking, λ_B as derived, is the average frequency of event B where $B = \{$one or more crashes during an independent subinterval of length $\tau_T\}$. We can write Eq. (8) as

$$\lambda = (\omega_T/2\pi) f(\theta) \exp(-h_o^2/2\sigma_e^2) \tag{19}$$

where $f(\theta) = \sigma_{e'}/(\omega_T \sigma_e)$ (see Table 1). Comparing Eqs. (18) and (19) we note that λ_B is independent of θ. Using τ_e instead of τ_T in deriving Eq. (18) would change this.

Figure 3 shows how the probability of crashing computed from expressions (6) and (14) for different h_o/σ_e ratios compare under the same set of conditions. It is evident that the missile would have to fly higher in order to have $P_C^{(A)}$ equal to $P_C^{(B)}$. Similar information is conveyed in an alternative form in Fig. 4 where $P_C^{(A)}$ and $P_C^{(B)}$ are plotted vs the number of terrain cycles in a particular mission. It will be noted that the value of $P_C^{(B)}$ would increase and be closer to $P_C^{(A)}$ under similar conditions if n were set equal to T/τ_e instead of T/τ_T, since, as previously discussed, $\tau_e < \tau_T$ in most practical cases.

References

[1] McHugh, G. C., Stoner, E. E., and Madonna, M. A., "Terrain Avoidance Maneuver Requirements Using a Radar Altimeter," Newsletter SCM-G&C-001, Sept. 6, 1972, McDonnell-Douglas, St. Louis, Mo.

[2] Weinstein, J. W., "Terrain Following Studies," TN GS-73-1, March 21, 1973, E Systems Inc.,

[3] Whittle, P., "On Stationary Processes in the Plane," *Biometrika*, Vol. 41, 1954, pp. 434–449.

[4] Webber, W. F., "On the Statistical Analysis of Random Surfaces," Ph.D. thesis, 1971, Southern Methodist University, Dallas, Texas.

[5] Newton, G. C., Gould, L. A., and Kaiser, J. F., *Analytical Design of Linear Feedback Controls*, Wiley, New York, 1957, Appendix E.

[6] Cunningham, E. P., "The Probability of Clobber for a Terrain-Following Missile," Internal Memo MCS-0-307, March 1, 1973, Applied Physics Lab., Silver Spring, Md.

[7] Papoulis, A., *Probability, Random Variables and Stochastic Processes*, McGraw-Hill, New York, 1965, Chap. 14.

[8] Beckman, P., *Probability in Communication Engineering*, Harcourt, Brace and World, New York, 1967, Chap. 6.

Particulate Infrared Radiation in Aluminized Solid-Fuel Rocket Plumes

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Introduction

THE far-field infrared emission of solid-propellant rocket exhaust at vacuum-expansion altitudes is of current interest, since such radiation may be an important factor in the detection of these vehicles. Those solid fuels which contain a significant amount of metallic aluminum release most of the metal into the exhaust in the form of oxide (Al_2O_3) particles, whose radius is on the order of microns.[1] These particles have temperatures on the order of the exit-plane gas temperature, and radiate significant amounts of thermal energy.

The formation mechanism of these particles has been treated,[2] as well as their role in engine impulse.[3] Effects of thermal radiation from the optically thick, near-field plume (close to the nozzle), impinging on the vehicle, have been studied.[4] Also, two-phase flow treatments of the motion of these particles in the exhaust have been performed.[5] Most recently, some effort has been made to predict the contribution of particulates to the far-field plume infrared signature of typical vehicles, based on various assumptions about the optical and thermal properties of the particles.[6] This present Note outlines a treatment of the general case of the radiant emission by particulate plumes at IR wavelengths. The parametric results of these computations are in a form applicable to the analysis of field data for the contribution of alumina particles to the total infrared emission of the plume, and for the optical and thermal properties of the particles themselves.

Analysis

The mean-mass particle radius lies in the range 4 to 8 μm for rockets with throat diameters above 5 in.[1] The particle size is of importance because, as predicted by the well-established Mie theory[7,8] emision at a given wavelength, λ, is not inhibited by geometric considerations only when $2\pi r/\lambda \gtrsim 1$, where r is the particle radius. For the infrared wavelengths of interest and the particle-size distributions encountered empirically, the inequality is satisfied.

The specific heat of pure aluminum oxide shows a temperature dependence.[9] The numerical calculations in this work were made assuming the Al_2O_3 in exhausts has the same temperature dependence as the pure substance. Also included in the calculations is the heat of fusion, ΔH_m, released at 2315°K, the melting temperature of Al_2O_3.

In the past, lab measurements of the emissivity of pure alumina in the shorter-wavelength infrared yielded values over the range of 10^{-4} to 10^{-2}. This is in contrast to the emissivities measured in aluminized rocket exhausts,[4] which are typically found to be much larger. Possibly because of polycrystalline or amorphous structure, impurities, and absorption bands at longer IR wavelengths, the exhaust particles exhibit average emissivities on the order of 0.25. Recently made lab measurements of the thermal properties of bulk alumina particles are more consistent with the higher emissivities of the exhaust measurements.[10]

It is important to note that the average emissivity, $\bar{\varepsilon}$, may differ from the spectral emissivity, ε_λ, at a given wavelength, λ. The spectral radiant energy emitted by a given mass of Al_2O_3

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